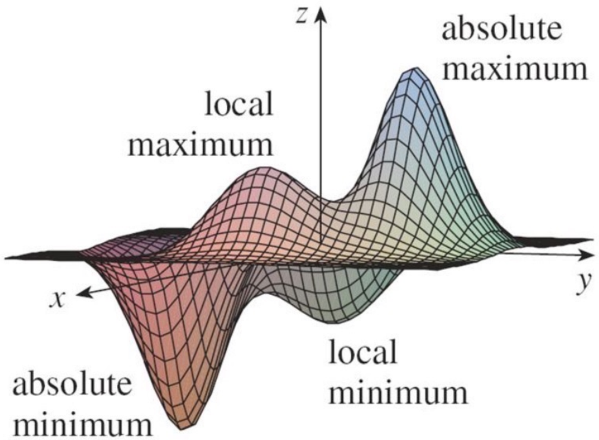
Review

Local extrema:



Thm: If f has a local maximum or minimum at (a,b) then

$$f_x(a,b)=0 \text{ and } f_y(a,b)=0.$$

Rmk: i.e. $\nabla f(a,b)=0$

Def: If $\nabla f(a,b)=0$, we call (a,b) a critical point of f.

Second derivatives test

Suppose (a,b) is a critical point of f (i.e. $f_x(a,b)=0$ and $f_y(a,b)=0$)

Set $\mathcal{D}=\mathcal{D}(a,b)=f_{xx}(a,b)f_{yy}(a,b)-(f_{xy}(a,b))^2$

(a) If $\mathcal{D}>0$ and $f_{xx}(a,b)>0$, then $f(a,b)$ - local min.;

(b) If $\mathcal{D}>0$ and $f_{xx}(a,b)<0$, then $f(a,b)$ - local max.;

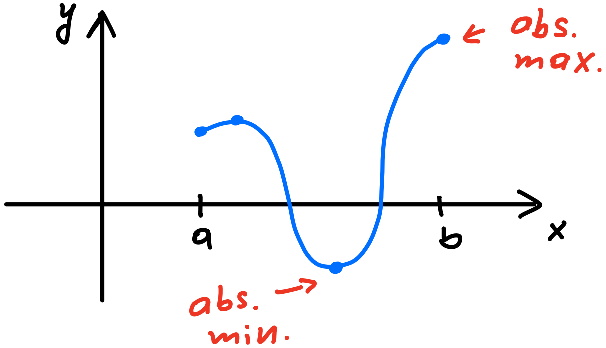
(c) If $\mathcal{D}<0$, then $f(a,b)$ is a saddle point of f.

Rmk: If $\mathcal{D}=0$, the test gives no information.

Rmk: $\mathcal{D}=\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy}-(f_{xy})^2$

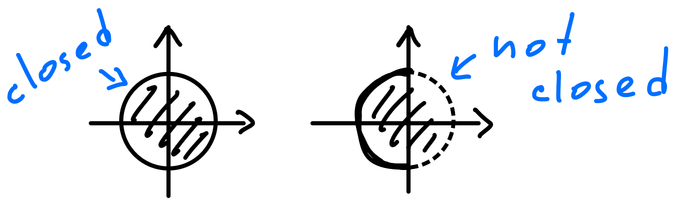
Extrema on bounded regions

Recall: $f(x)$ continuous on $[a,b]$, has an absolute max. and an abs. min. on $[a,b]$.



- We find them by evaluating f on critical points AND on endpoints.

$\mathcal{D}$ - closed set in $\mathbb{R}^2$ (i.e. contains boundary pts); bounded set (contained in some disk)



f, continuous on a closed, bounded $\mathcal{D} \subset \mathbb{R}^2$, attains an abs. max. value $f(x_1, y_1)$ and an abs. min. value $f(x_2, y_2)$ at some points in $\mathcal{D}$.

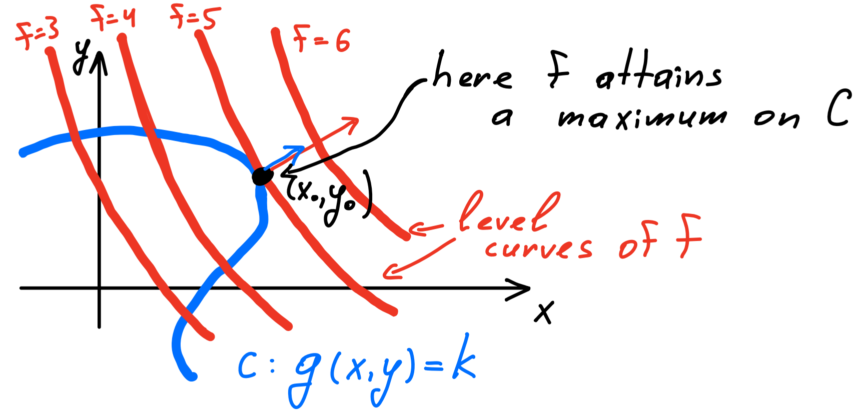
To find abs. max/min values of f on a closed, bounded $\mathcal{D}$:

1. Find values of f at critical points in $\mathcal{D}$;
2. Find extreme values on the boundary of $\mathcal{D}$;
3. largest of 1, 2 – abs. max.
   smallest of 1, 2 – abs. min.

Lagrange multipliers (one constrain)

Problem:

Find extreme values of $f(x,y)$ subject to constraint $g(x,y)=k$



level curves $g(x,y)=k$ and $f(x,y)=5$ touch at (x_0,y_0)

$$\Rightarrow \nabla f(x_0,y_0) = \underbrace{\lambda}_{} \nabla g(x_0,y_0)$$

↑ a number, "Lagrange multiplier"

Method of Lagrange multipliers

To find maximum & minimum values of $f(x,y)$ subject to constrain $g(x,y)=k$:

ⓐ find all values of x, y, λ such that

- $\nabla f(x,y) = \lambda \nabla g(x,y)$ ← $\boxed{\begin{array}{l} f_x = \lambda g_x \\ f_y = \lambda g_y \end{array}}$
- $g(x,y) = k$

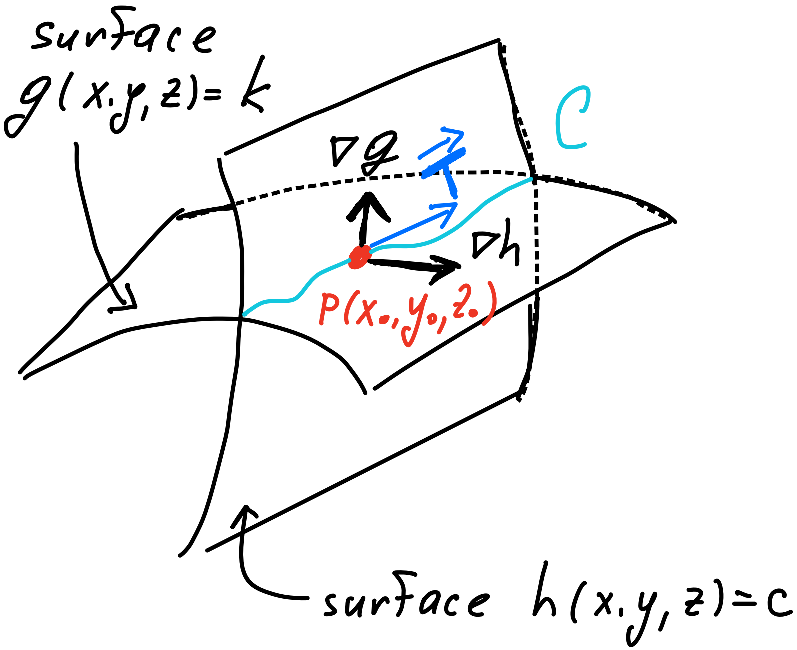
ⓑ evaluate f at all points found in (a)

maximum = largest value of f

minimum = smallest

Lagrange multipliers with two constraints

Problem: Find min/max values of $f(x,y,z)$ subject to two constraints: $\begin{cases} g(x,y,z) = k \\ h(x,y,z) = c \end{cases}$ define a curve C



> If at $P(x_0, y_0, z_0)$ on C, f has a min or max, then directional derivative
>
> $$D_{\vec{T}} f = 0, \qquad (D_{\vec{T}} f = \nabla f \cdot \vec{T})$$
>
> where $\vec{T}$ is the unit tangent vector along C at P.

So, $\nabla f(x_0, y_0, z_0)$ is orthogonal to $\vec{T}$

$\Rightarrow \nabla f|_P$ lies in the plane orthogonal to C at P.

This plane is defined by $\nabla g(x_0, y_0, z_0)$ and $\nabla h(x_0, y_0, z_0)$.

Therefore

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)$$

Two Lagrange multipliers

Method of Lagrange multipliers

To find maximum & minimum values of $f(x,y,z)$
subject to constraints $g(x,y,z)=k$ and $h(x,y,z)=c$

ⓐ find all values of x, y, z, λ, μ such that

- $\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z)$
- $g(x,y,z) = k$
- $h(x,y,z) = c$

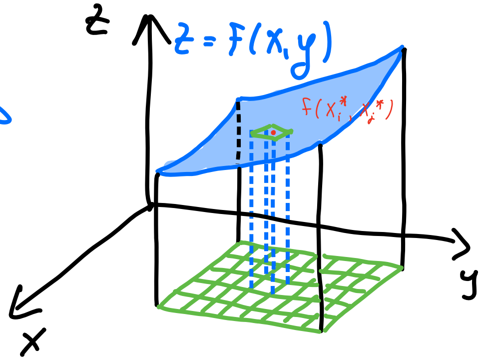
$$\boxed{\begin{aligned} f_x &= \lambda g_x + \mu h_x \\ f_y &= \lambda g_y + \mu h_y \\ f_z &= \lambda g_z + \mu h_z \end{aligned}}$$

ⓑ evaluate f at all points found in (a)

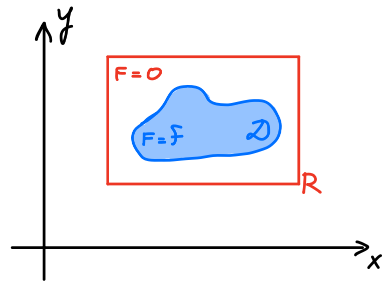
maximum = largest value of f
minimum = smallest

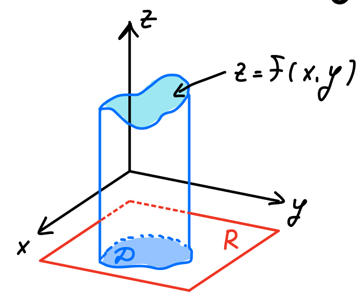
Double integrals:

$$\iint_R f(x,y)\,dA = \lim_{\substack{m\to\infty\\ n\to\infty}} \sum_{i,j} f(x_i^*, y_j^*)\underbrace{\Delta A}_{\Delta x\cdot\Delta y} \overset{\text{for } f\geq 0}{=} \text{volume of a solid that lies above the rectangle (general region) and below the surface } z=f(x,y).$$



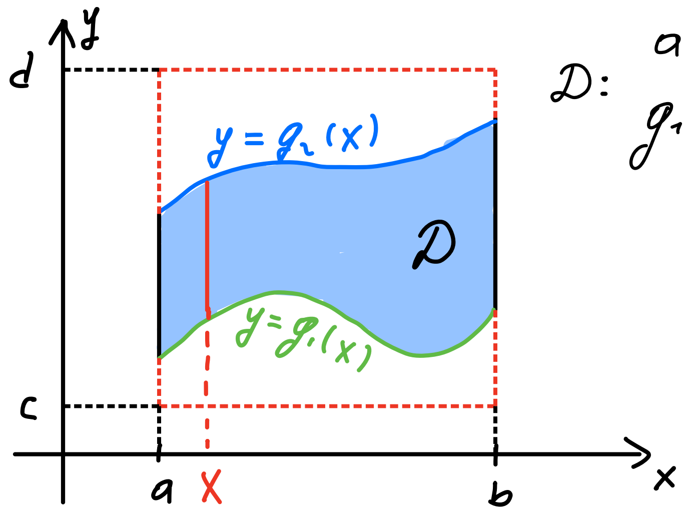
(x_i^*, y_j^*) - sample point in $[x_{i-1},x_i]\times[y_{j-1},y_j]$





Fubini's thm: if $f(x,y)$ continuous

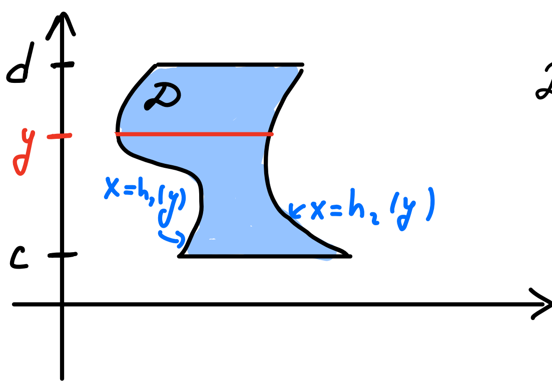
$$\iint_R f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy$$



$$D: \begin{array}{l} a\leq x\leq b \\ g_1(x)\leq y\leq g_2(x)\end{array}$$

region of type I

$$\boxed{\iint_D f(x,y)\,dA = \int_a^b\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx}$$



$$D: \begin{array}{l} c\leq y\leq d \\ h_1(y)\leq x\leq h_2(y)\end{array}$$

region of type II

$$\boxed{\iint_D f(x,y)\,dA = \int_c^d\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy}$$

True-False Quiz

True-False Quiz:

1) If a function has a local maximum at a point, then its gradient must be zero at that point.

2) To find the absolute max/min of a continuous function on a closed and bounded region, it is enough to check only the critical points inside the region.

3) The method of Lagrange multipliers can find both maxima and minima on a bounded region.

4) If $\nabla f = \lambda \nabla g$ at a point, then the value at that point must be max or min of f under the constraint $g(x,y)=0$.

5) $$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx = \int_{g_1(x)}^{g_2(x)} \int_a^b f(x,y)\,dx\,dy$$

6) A double integral always gives the volume under the surface $z=f(x,y)$.

7) Let D be any region in the plane whose projection onto the x-axis is $[a,b]$. Then $$\iint_D (x^2+e^x)\,dA = \int_a^b (x^2+e^x)\cdot(\text{length of the vertical slice of } D \text{ at } x)\,dx.$$

Quiz Answers

True-False Quiz: 1) If a function has a local maximum at a point, then its gradient must be zero at that point. $\boxed{\text{TRUE}}$

2) To find the absolute max/min of a continuous function on a closed and bounded region, it is enough to check only the critical points inside the region. $\boxed{\text{FALSE}}$

(must also check the boundary)

3) The method of Lagrange multipliers can find both maxima and minima ~~on a bounded region~~. $\boxed{\text{FALSE}}$

subject to a constraint (which can be seen as a bdry of some region)

The method of Lagrange multipliers can be used to find max/min on the BOUNDARY of some region, but does NOT give information about the points inside the region.

4) If $\nabla f = \lambda \nabla g$ at a point, then the value at that point must be max or min of f under the constraint $g(x,y)=0$. $\boxed{\text{FALSE}}$

It also could be a saddle point.

5) $$\underbrace{\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx}_{\text{this is a number}} \neq \underbrace{\int_{g_1(x)}^{g_2(x)} \int_a^b f(x,y)\,dx\,dy}_{\text{this is some function } A(x)}$$

$\boxed{\text{FALSE}}$

6) A double integral always gives the volume under the surface $z = f(x,y)$.

$\boxed{\text{FALSE}}$

This is true only when $f(x,y) \geq 0$.

7) Let $\mathcal{D}$ be any region in the plane whose projection onto the x-axis is $[a,b]$.

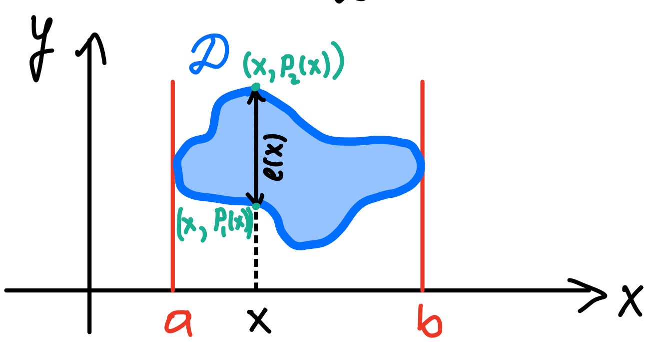
Then $$\iint_{\mathcal{D}} f(x)\,dA = \int_a^b f(x)\cdot(\text{length of the vertical slice of } \mathcal{D} \text{ at } x)\,dx.$$

$\boxed{\text{TRUE}}$

Indeed, $$\iint_{\mathcal{D}} f(x)\,dA = \int_a^b \underbrace{\int_{P_1(x)}^{P_2(x)} f(x)\,dy}\,dx = \int_a^b f(x)\,\ell(x)\,dx$$

$$y\,f(x)\Big|_{P_1(x)}^{P_2(x)} = (P_2(x) - P_1(x))\,f(x) = \ell(x)\,f(x)$$

// since $f(x)$ does NOT depend on y //



Rmk: It is important that we integrate $f(x)$ here NOT $f(x,y)$.

Few more problems

1) Find the volume of the largest rectangular box in the 1^{st} octant, with faces parallel to xy, yz, xz-planes and one vertex in the plane $x + 2y + 3z = 6$.

2) Sketch the region D and rewrite $\iint\limits_{D} f(x,y)\,dA$ as

$$\int_{?}^{?}\int_{?}^{?} f(x,y)\,dx\,dy \quad \text{or} \quad \int_{?}^{?}\int_{?}^{?} f(x,y)\,dy\,dx,$$

where the region D is given by

a) $D = \{(x,y) \mid 1 \le x \le 2,\ 0 \le y \le \sqrt{3-x}\}$

b) $D = \{(x,y) \mid 0 \le x \le \pi,\ 0 \le y \le \sin x\}$

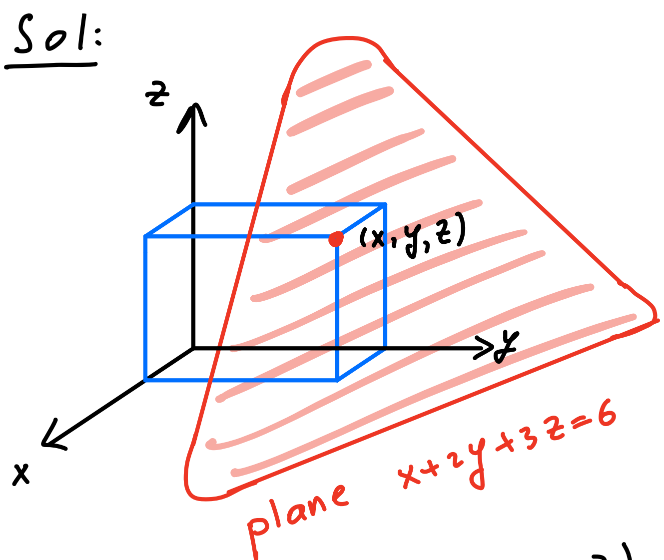
c) $D = \{(x,y) \mid 1 \le x \le 6,\ 14 + x \le 5y \le 11 + 4x\}$

d) $D = \{(x,y) \mid 1 \le x \le 2,\ 2 - x \le y \le \sqrt{1-(x-1)^2}\}$

Solutions

(1) Find the volume of the largest rectangular box in the 1^{st} octant, with faces parallel to xy, yz, xz-planes and one vertex in the plane $x+2y+3z=6$.

Sol:

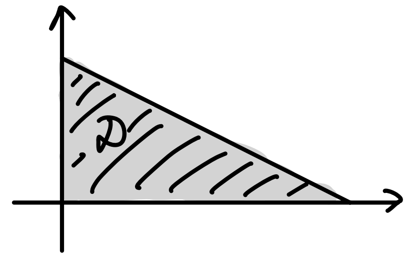


1) $z = \dfrac{6-x-2y}{3}$ $\qquad$ Volume $= x \cdot y \cdot z$

$$\text{Volume} = f(x,y) = xy\,\frac{6-x-2y}{3}$$

We want to maximize f in

$$D = \left\{ (x,y) : \begin{matrix} x+2y \le 6 \\ x \ge 0,\ y \ge 0 \end{matrix} \right\}$$



2) $f_x(x,y) = y\,\dfrac{6-x-2y}{3} - \dfrac{1}{3}xy$

$f_y(x,y) = x\,\dfrac{6-x-2y}{3} - \dfrac{2}{3}xy$

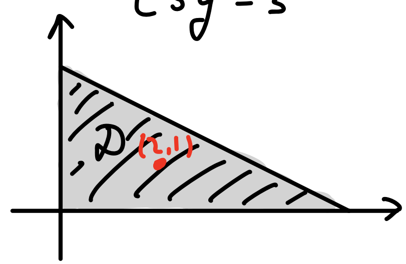
$$\begin{cases} y\,\dfrac{6-2x-2y}{3} = 0 \\ x\,\dfrac{6-x-4y}{3} = 0 \end{cases}$$

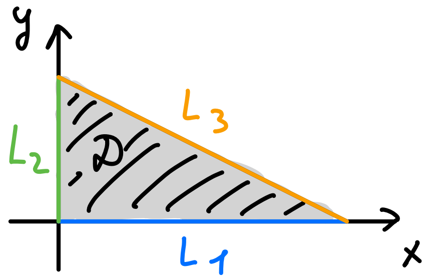
$x=0$ or $y=0$ $\leftarrow$ on the boundary of D

$$\begin{cases} 6-2x-2y=0 \\ 6-x-4y=0 \end{cases} \iff \begin{cases} x+y=3 \\ x+4y=6 \end{cases} \qquad \begin{cases} x=3-y \\ 3y=3 \end{cases}$$

$(2,1)$ - critical point in the interior of D

$$f(2,1) = 2 \cdot 1 \cdot \frac{6-2-2\cdot 1}{3} = \frac{4}{3}$$





On L_1: $F(x,0)=0 \quad , \quad 0 \le x \le 6$

On L_2: $F(0,y)=0 \quad , \quad 0 \le y \le 3$

On L_3: $0 \le y \le 3 \qquad x = 6-2y$
$x+2y=6$

$$F(x,y) = xy\,\frac{6-x-2y}{3} = (6-2y)\,y\,\frac{\overbrace{6-(6-2y)-2y}^{=0}}{3} = 0$$

$$\boxed{\text{Abs. max.: } F(2,1)=\tfrac{4}{3}\ ,\ \text{box of size } \underset{x}{2}\times\underset{y}{1}\times\underset{z}{\tfrac{2}{3}}}$$

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(2) Sketch the region $\mathcal{D}$ and rewrite $\iint\limits_{\mathcal{D}} f(x,y)\,dA$ as
$\int_{\boxed{?}}^{\boxed{?}}\int_{\boxed{?}}^{\boxed{?}} f(x,y)\,dx\,dy$ or $\int_{\boxed{?}}^{\boxed{?}}\int_{\boxed{?}}^{\boxed{?}} f(x,y)\,dy\,dx$, where $\mathcal{D}$ is given by

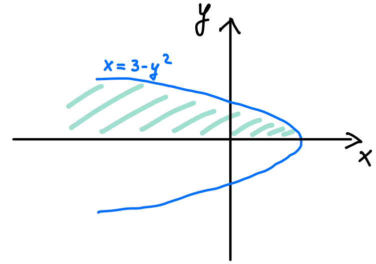
a) $\mathcal{D} = \{(x,y) \mid 1 \le x \le 2,\ 0 \le y \le \sqrt{3-x}\}$

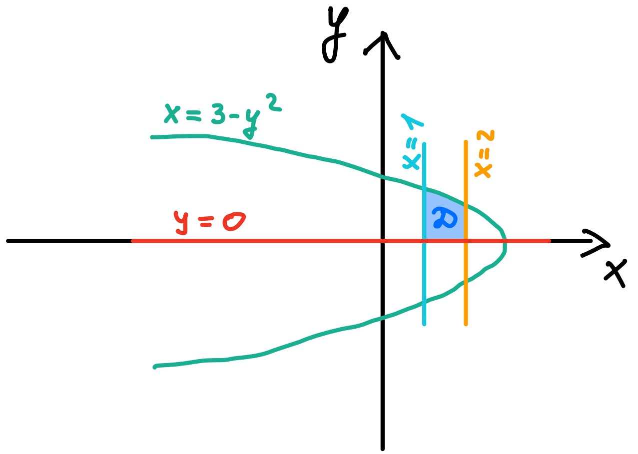
b) $\mathcal{D} = \{(x,y) \mid 0 \le x \le \pi,\ 0 \le y \le \sin x\}$

c) $\mathcal{D} = \{(x,y) \mid 1 \le x \le 6,\ 14+x \le 5y \le 11+4x\}$

d) $\mathcal{D} = \{(x,y) \mid 1 \le x \le 2,\ 2-x \le y \le \sqrt{1-(x-1)^2}\}$

Sol: **(2a)** $0 \le y \le \sqrt{3-x} \iff \boxed{\begin{cases} 0 \le y \\ y^2 \le 3-x \end{cases}}$



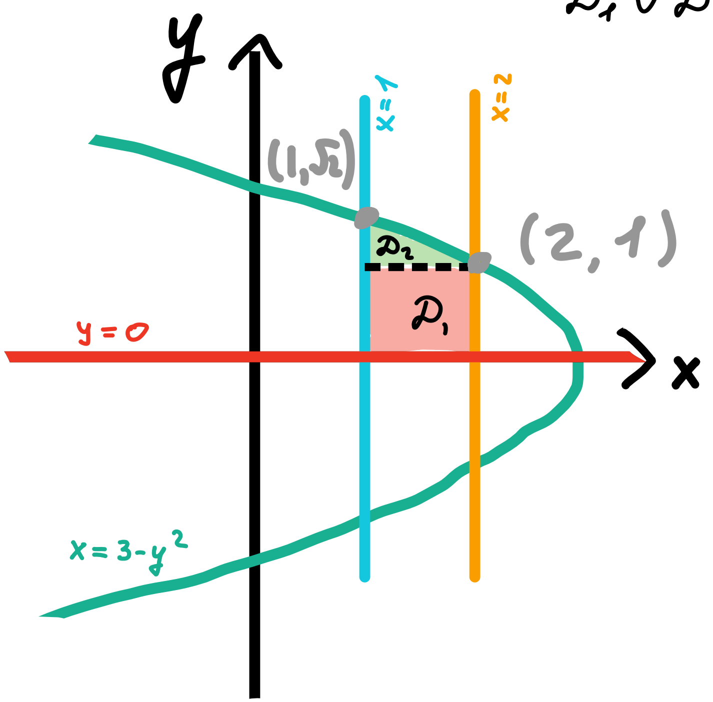


$$\iint_{D} f(x,y)\,dA = \int_1^2 \int_0^{\sqrt{3-x}} f(x,y)\,dy\,dx \;\boxed{=}$$

$$\underline{\int_0^1 \int_1^2 f(x,y)\,dx\,dy} + \underline{\int_1^{\sqrt{2}} \int_1^{3-y^2} f(x,y)\,dx\,dy}$$

$$= \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$

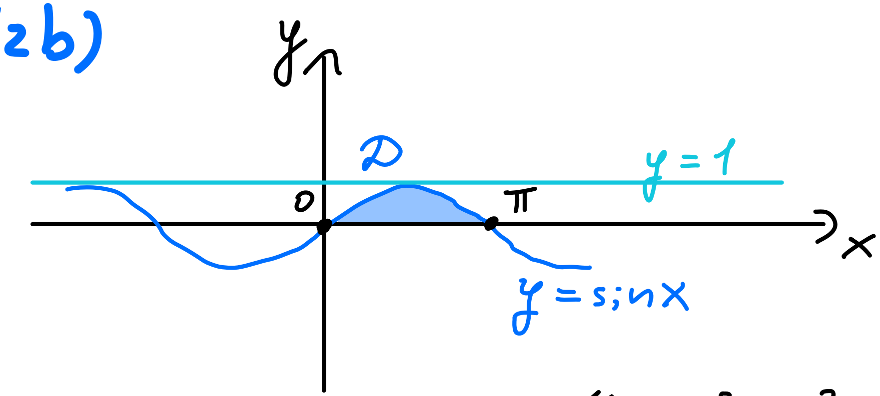
$D_1 \cup D_2 = D$



// $\begin{cases} x = 2 \\ x = 3-y^2 \\ y \ge 0 \end{cases} \Rightarrow \begin{cases} x = 2 \\ y = 1 \end{cases}$

$\begin{cases} x = 1 \\ x = 3-y^2 \\ y \ge 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = \sqrt{2} \end{cases}$ //

(2b)

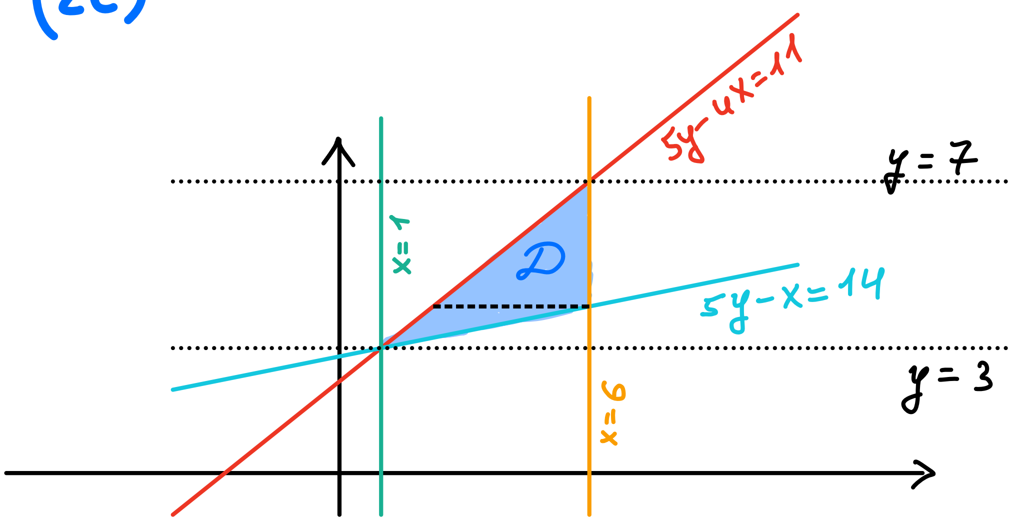


$$\iint_D f(x,y)\,dA = \int_0^{\pi} \int_0^{\sin x} f(x,y)\,dy\,dx$$

$$= \int_0^1 \int_{\arcsin(y)}^{\pi - \arcsin(y)} f(x,y)\,dx\,dy$$

// On $[0,\pi]$ $\sin x = y$ at $x = \arcsin(y)$ and $x = \pi - \arcsin(y)$ //

(2c)



$$\iint_D f(x,y)\,dA = \int_1^6 \int_{\frac{14+x}{5}}^{\frac{11+4x}{5}} f(x,y)\,dy\,dx$$

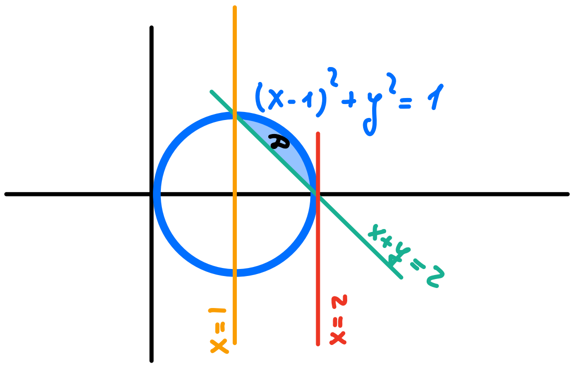
$$= \int_3^4 \int_{\frac{5y-11}{4}}^{5y-14} f(x,y)\,dx\,dy + \int_4^7 \int_{\frac{5y-11}{4}}^{6} f(x,y)\,dx\,dy$$

(2d) $\mathcal{D} = \{ (x,y) \mid 1 \le x \le 2,\ 2-x \le y \le \sqrt{1-(x-1)^2} \}$

$1 \le x \le 2 \implies 2-x \ge 0 \implies y \ge 0$

$\begin{cases} y \le \sqrt{1-(x-1)^2} \\ y \ge 0 \end{cases} \iff \begin{cases} y^2 \le 1-(x-1)^2 \\ y \ge 0 \end{cases} \iff \begin{cases} y^2 + (x-1)^2 \le 1 \\ y \ge 0 \end{cases}$

$2-x \le y \implies x+y \ge 2$



$$\iint_{\mathcal{D}} f(x,y)\,dA = \int_1^2 \int_{2-x}^{\sqrt{1-(x-1)^2}} f(x,y)\,dy\,dx$$

// $(x-1)^2 + y^2 \le 1$

$\implies (x-1)^2 \le 1-y^2$

$\implies x-1 \le \sqrt{1-y^2}$ //

$$= \int_0^1 \int_{2-y}^{1+\sqrt{1-y^2}} f(x,y)\,dx\,dy$$